On Michael Berry's Model of Band-limited

Superoscillatory Functions

Nicholas Wheeler
February 2018

Introduction. It was in 1990 conversation with Aharonov (then visiting the University of Bristol, from which he had, as a student of David Bohm, taken his PhD in 1960) that Michael Berry first learned of the counterintuitive phenomenon to which he gave the name "superoscillation." To a conference convened to celebrate Aharonov's 60^{th} birthday (University of South Carolina, 1992) Berry contributed a paper[1] in which he described and undertook to develop properties of a continuous generalization—actually, a class of continuous generalizations—of what I call "Aharonov's construction."[2]

For days I have been struggling to understand Berry's paper, particularly as it relates to the consequences of band-limiting. Tired of wandering in circles, of gaining occasional insights only to lose them again in the underbrush of many *Mathematica* notebooks, my effort here will be to subject my present muddle of understanding/confusion/ignorance to the discipline of the written page.

Aharonov's construction. The "superoscillation phenomenon" discovered by Yakir Aharonov is most easily exemplified by what I call "Aharonov's function"

$$\mathbb{A}(x,a,n) = \big[\cos\left(\tfrac{x}{n}\right) + i\,a\sin\left(\tfrac{x}{n}\right)\big]^n \tag{1}$$

which when subjected to the process $n\to\infty$ becomes "Aharonov's construction." For large n we in leading order have

$$\approx \big[1 + \tfrac{1}{n}iax\big]^n \longrightarrow e^{iax}$$

At $a = 1$ we have $\mathbb{A}(x,1,n) = [\exp\left(i\frac{x}{n}\right)]^n = e^{ix}$ trivially, for all n; it is in the cases $a \neq 1$, and more specifically in the cases $a > 1$, that (1) acquires interest, for by the binomial theorem (1) can be developed

$$\mathbb{A}(x,a,n) = \sum_{p=0}^{n} A(p,a,n)e^{ik(p,n)x} \tag{2}$$
$$A(p,a,n) = \tfrac{1}{2^n}\tbinom{n}{p}(1+a)^{n-p}(1-a)^p$$
$$k(p,n) = 1 - 2p/n$$

[1] "Faster than Fourier," published in the proceedings of that conference, *Quantum Coherence and Reality: In Celebration of the* 60^{th} *Birthday of Yakir Aharonov* (1994), available on the web.

[2] Actually it was—if I read Berry correctly—in continuous language that Aharonov sketched his discovery in that initial conversation with Berry.

as a weighted sum of $n+1$ complex exponentials ("Fourier terms") with wave numbers $k(p,n) \in \{1, 1-\frac{2}{n}, \dots, -1+\frac{2}{n}, -1\}$ that equi-partition the interval $[-1,+1]$, meaning that all wavelengths $\lambda(p,n) \geqslant 2\pi$. What makes $\mathbb{A}(x,p,n)$ so remarkable is the fact that for any given $a > 1$ the construction

$$\mathbb{A}(x,p,n\to\infty) = e^{iax}$$

has *wavenumber larger—and wavelength shorter*—than any of the Fourier terms that appear on the right side of (2).

I have discussed properties of Aharonov's function and provided graphic illustration of the superoscillatory phenomenon in a pair of recent notes.[3] Here I mention only that the complex-valued function $\mathbb{A}(x,a,n)$ (i) is periodic

$$\mathbb{A}(x,a,n) = \mathbb{A}(x+2\pi n,a,n)$$

(ii) assumes unit value at the origin $\mathbb{A}(0,a,n)=1$ and periodically thereafter; (iii) approximates its asymptote e^{iax} only in $\{a,n\}$-dependent neighborhoods of those points, and (iv) that the situation is actually a bit more complicated than those remarks suggest. For

$$F(x,a,n) \equiv |\mathbb{A}(x,a,n)| = \left[\cos^2\left(\tfrac{x}{n}\right) + a^2\sin^2\left(\tfrac{x}{n}\right)\right]^{n/2}$$

has half the period of $\mathbb{A}(x,a,n)$:

$$F(x,a,n) = F(x+\pi n,a,n)$$

We see that

$$F_{\max} = a^n$$

is achieved *twice* per $\mathbb{A}$-period. We note that as $n\to\infty$ the period $2\pi n$ increases without bound, and so (assuming always that $a>1$) does a^n.

The superoscillation phenomenon is exposed most vividly (as demonstrated in a portfolio of figures[3]) when n is large but finite, when they appear in an x-interval of characteristic width $\sqrt{n}/a$, centered on the troughs of the function $F(x,a,n)$.

Barry's construction. The objective of Barry's program can be symbolized

$$\lim_{n\to\infty}\sum_{p=0}^{n} \longrightarrow \lim_{\nu\to\infty}\int dp$$

To that end, he writes

[3] "When the whole vibrates faster than any of its parts: Computational superoscillation theory," (December, 2017); "A note concerning the length of Aharonov's superoscillatory interval," (January, 2018).

$$\mathbb{B}(x,b,\nu) = \int_{-\infty}^{\infty} B(p,b,\nu)\, e^{ixk(p)} dp \tag{3}$$

where $k(p)$ is a real-valued function of the real variable p and ranges on $[-1,+1]$. He requires moreover that it be a property of $B(p,b,\nu)$ that

$$\lim_{\nu\to\infty} B(p,b,\nu) = \delta(p-ib) \tag{4}$$

where ν is a continuous analog of the discrete parameter n. Then

$$\lim_{\nu\to\infty} \mathbb{B}(x,b,\nu) = e^{ixk(ib)} = e^{iax} \quad \text{with} \quad a = k(ib) \tag{5}$$

He imposes upon $k(p)$ additional requirements intended to ensure that $a = k(ib)$ is real and $\geqslant 1$. Specifically, he requires that $k(0) = 1$, that $k(p+iq)$ assumes only real values on the imaginary axis ($p = 0$) in the complex p-plane (this is ensured if $k(p)$ is a function of p^2), and that $a = k(ib)$ ranges $1 \to \infty$ as b ranges $0 \to b_{\text{max}}$. Barry lists several functions $k(p)$ with the requisite properties, among them

$$\begin{aligned} k_1(p) &= \frac{1}{1+\frac{1}{2}p^2} &&: \quad b_{\text{max}} = \sqrt{2} \\ k_2(p) &= \operatorname{sech} p &&: \quad b_{\text{max}} = \pi/2 \\ k_3(p) &= \exp\big\{-\tfrac{1}{2}p^2\big\} &&: \quad b_{\text{max}} = \infty \end{aligned}$$

To achieve (4) one looks to representations of the δ-function, of which the literature supplies many; among those most commonly encountered are

$$\begin{aligned} D_1(p-p_0,\nu) &= \sqrt{\nu/\pi}\exp\big\{-\nu(p-p_0)^2\big\} \\ D_2(p-p_0,\nu) &= \tfrac{1}{2}\nu\operatorname{sech}^2\big\{\nu(p-p_0)\big\} \end{aligned}$$

both of which yield $\delta(p-p_0)$ in the limit $\nu\to\infty$. It is the tractability of the integral (3) that dictates the optimal selection. Barry elects to work with the Gaussian representation $D_1(p-p_0,\nu)$, which brings (3) to the form

$$\mathbb{B}_\kappa(x,b,\nu) = \sqrt{\nu/\pi}\int_{-\infty}^{\infty} \exp\big\{-\nu(p-ib)^2 + ixk_\kappa(p)\big\} dp \tag{6.1}$$

where

$$-\nu(p-ib)^2 + ixk_\kappa(p) = \nu\big[-\big(p^2-b^2\big) + i\,2\big(\xi\, k_\kappa(p) + bp\big)\big] \tag{6.2}$$

with $\xi = x/2\nu$ and $\kappa = 1,2,3$. So far as I am aware, none of the integrals (6.1) can be developed in analytic closed form (except—by design—in the limit $\nu\to\infty$, where one has (5)). But if one introduces (6.2) into (6.1) one obtains an integral of the form

$$\int_{-\infty}^{\infty} e^{\,\nu f(p\,;\,b,\,\xi)} dp$$

for which one can obtain asymptotic approximations by the method of steepest descent. The effort then becomes one of discovering the locations (on the complex p-plane) of the saddle points of the analytic function $f(p;b,\xi)$. That is a problem which I have explored in a series of *Mathematica* notebooks, with (remarkably complicated) results that I intend to summarize in a separate essay. But my present interest lies elsewhere.

Barry's model of band-limited superoscillation. Each of the functions $k_\kappa(p)$ can be developed

$$k_\kappa(p) = 1 - \tfrac{1}{2}p^2 + \text{terms of higher even order}$$

The function

$$k(p) = 1 - \tfrac{1}{2}p^2 \tag{7}$$

is so simple that the integral (6.1)—now effectively Gaussian—can be evaluated explicitly:

$$\begin{aligned}
\mathbb{B}(x,b,\nu) &= \sqrt{\nu/\pi}\int_{-\infty}^{\infty}\exp\big\{-\nu(p-ib)^2 + ixk(p)\big\}dp\\
&= \frac{1}{\sqrt{1+i\xi}}\exp\Big[ix\Big(1+\frac{\frac{1}{2}b^2}{1+i\xi}\Big)\Big]\\
&= \frac{1}{\sqrt{1+i\xi}}\exp\Big[\frac{b^2x\xi}{2(1+\xi^2)}\Big]\cdot\exp\Big[ix\Big(1+\frac{\frac{1}{2}b^2}{(1+\xi^2)}\Big)\Big]
\end{aligned}\tag{8}$$

where the last step was accomplished by the command `ComplexExpand` and where again $\xi = x/2\nu$. Immediately,

$$\lim_{\nu\to\infty}\mathbb{B}(x,b,\nu) = \exp\big[ix\big(1+\tfrac{1}{2}b^2\big)\big] = e^{ixk(ib)}$$

as was anticipated at (5).

We pause to look to the detailed implications of (8). The

$$\text{oscillatory factor} = e^{ixK(b,\xi)}$$

where

$$K(b,\xi) = \Big(1+\frac{\frac{1}{2}b^2}{(1+\xi^2)}\Big) \quad : \quad \begin{cases} >1 \text{ for all finite } b,\xi\\ \approx k(ib) \text{ for } \xi\approx 0;\ \textit{i.e.}, \text{ for } x\ll\nu\\ \approx 1 \text{ for } \xi\gg 1;\ \textit{i.e.}, \text{ for } x\gg\nu\\ \text{is superoscillatory for all } b\text{-values}\end{cases}$$

The

$$\text{prefactor} = \text{amplitude}\ e^{i\,\text{phase}}$$

where (use `ComplexExpand[`$1/\sqrt{1+i\xi}$`]`)

$$\begin{aligned}
\text{amplitude} &= \frac{1}{(1+\xi^2)^{\frac{1}{4}}}\exp\Big[\frac{b^2x\xi}{2(1+\xi^2)}\Big] \quad : \quad \begin{cases}\approx 1 \text{ for } \xi\approx 0;\ \textit{i.e.}, \text{ for } x\ll\nu\\ \approx 0 \text{ for } \xi\gg 1\end{cases}\\
\text{phase} &= -\tfrac{1}{2}\texttt{Arg}[1+i\xi]\\
&= -\tfrac{1}{2}\arctan\xi \quad : \quad \begin{cases}\approx 0 \text{ for } \xi\approx 0\\ \approx -\tfrac{1}{4}\pi \text{ for } \xi\gg 1, \text{ after rapid descent}\end{cases}
\end{aligned}$$

These results are, however, subject to a fundamental criticism: they proceed from an integral that describes a weighted superposition of e^{ikx}-terms with wavenumbers $k(p)$ that range from $k(0) = 1$ all the way down to $k(\pm\infty) = -\infty$. Such terms possess all possible wavelengths, so have nothing to say about the superoscillation phenomenon. The function $k(p)$ falls, as required, within the interval $[-1, +1]$ only if $-2 \leqslant p \leqslant +2$, so if we are to achieve evidence of "band-limited superoscillation" we must restrict the limits of integration. Berry drew attention to the happy fact that the resulting integral, as a Gaussian integral between finite limits, still admits of exact analytic evaluation; *Mathematica* supplies a result that can after simplifications[4] be written

$$\mathbb{B}_{\text{band-limited}}(x, b, \nu) = \sqrt{\nu/\pi}\int_{-2}^{2} \exp\left\{-\nu(p - ib)^2 + ixk(p)\right\}dp \tag{9.0}$$

$$= \mathbb{B}(x, b, \nu) \times \mathcal{E}(x, b, \nu) \tag{9.1}$$

with

$$\mathcal{E}(x, b, \nu) = \tfrac{1}{2}\text{erf}\Big[2\sqrt{\nu}\,\frac{1 + i\xi + \frac{1}{2}ib}{\sqrt{1 + i\xi}}\Big] + \tfrac{1}{2}\text{erf}\Big[2\sqrt{\nu}\,\frac{1 + i\xi - \frac{1}{2}ib}{\sqrt{1 + i\xi}}\Big] \tag{9.2}$$

For fixed x the parameter $\xi = x/2\nu \to 0$ asymptotically (*i.e.*, as $\nu \to \infty$), and at $\xi = 0$ we have

$$\mathcal{E}(x, b, \nu)\Big|_{\xi=0} = \tfrac{1}{2}\text{erf}\big[2\sqrt{\nu}\,(1 + \tfrac{1}{2}ib)\big] + \tfrac{1}{2}\text{erf}\big[2\sqrt{\nu}\,(1 - \tfrac{1}{2}ib)\big]$$

which by $\text{erf}(\bar{z}) = \overline{\text{erf}(z)}$ is real, and by

$$\lim_{x\to\infty} \text{erf}(x + iy) = 1$$

assumes asymptotically the value 1. So asymptotically (9) gives back (8).

To describe the values assumed by $\mathbb{B}_{\text{band-limited}}(x, \nu, b)$ when $0 < \nu < \infty$ one must look to properties of the analytic function $\text{erf}(x + iy)$, and it is in this connection that the story becomes much more interesting. Figure 1 shows the null curves of the real and imaginary parts of $\text{erf}(x + iy)$. Figures 2 provide a refined display of that same information, which Figures 3 display in 3D detail. In the 1$^{\text{st}}$, 4$^{\text{th}}$, 5$^{\text{th}}$ and 8$^{\text{th}}$ octants the real part of $\text{erf}(x + iy)$ never deviates by more than a few parts in a thousand from the value 1, while its imaginary part remains similarly close to the value 0. In the other octants both parts foliate and assume very large positive/negative values. The foliations interdigitate in such a way as to produce a smooth absolute value (Figure 4); the rapidity of its ascent in the 2$^{\text{nd}}$, 3$^{\text{rd}}$, 6$^{\text{th}}$ and 7$^{\text{th}}$ octants is demonstrated in Figure 5, where 10-fold increases in equally separated, indicating exponential growth.

---

[4] Use `ComplexExpand` to obtain $\frac{1}{2} \pm \frac{i}{2} = \sqrt{\pm i/2}$.

Looking now to how those properties of $\text{erf}(x+iy)$ relate to properties of the $\mathcal{E}(x,b,\nu)$ of (9.2), whence via (9.1) to the $\mathbb{B}_{\text{band-limited}}(x,b,\nu)$ of (9.0), we write

$$2\sqrt{\nu}\,\frac{1+i\xi+\frac{1}{2}ib}{\sqrt{1+i\xi}}=X(x,\nu,b)+i\,Y(x,\nu,b)$$

with

$$\begin{aligned}X(x,\nu,b)&=\frac{2\sqrt{\nu}}{(1+\xi^2)^{\frac{1}{4}}}\Big[\big(\xi+\tfrac{1}{2}b\big)\sin\big(\tfrac{1}{2}\text{arctan}\,\xi\big)+\cos\big(\tfrac{1}{2}\text{arctan}\,\xi\big)\Big]\\Y(x,\nu,b)&=\frac{2\sqrt{\nu}}{(1+\xi^2)^{\frac{1}{4}}}\Big[\big(\xi+\tfrac{1}{2}b\big)\cos\big(\tfrac{1}{2}\text{arctan}\,\xi\big)-\sin\big(\tfrac{1}{2}\text{arctan}\,\xi\big)\Big]\\Z(x,\nu,b)&=X(x,\nu,b)+i\,Y(x,\nu,b)\end{aligned}$$

in which we have again employed the abbreviation $\xi=x/2\nu$. In this detailed notation (9.2) reads

$$\mathcal{E}(x,\nu,b)=\tfrac{1}{2}\text{erf}\big(Z(x,\nu,b)\big)+\tfrac{1}{2}\text{erf}\big(Z(x,\nu,-b)\big)$$

In Figures 6 the functions $\mathcal{E}(2,50,b)$ and $\mathcal{E}(3,50,b)$ are seen to maintain unit value up to the near-neighborhood of $b=2$, where they abruptly begin rapid large-amplitude oscillation, as these results demonstrate numerically:

$$\begin{aligned}\mathcal{E}(2,50,1.9)&=1.0000-i\,7.1789\times10^{-11}\\\mathcal{E}(3,50,1.9)&=1.0000+i\,6.2293\times10^{-11}\end{aligned}$$

$$\begin{aligned}\mathcal{E}(2,50,2.1)&=+1.0484\times10^{7}+i\,1.6892\times10^{7}\\\mathcal{E}(3,50,2.1)&=-1.7794\times10^{7}+i\,7.9214\times10^{5}\end{aligned}$$

The complex coordinates of those points are

$$\begin{aligned}Z(2,50,1.9)&=14.2772+i\,13.5744\\Z(2,50,2.1)&=14.2913+i\,14.9884\end{aligned}\tag{10.1}$$

$$\begin{aligned}Z(3,50,1.9)&=14.3451+i\,13.6426\\Z(3,50,2.1)&=14.3663+i\,15.0563\end{aligned}\tag{10.2}$$

all of which fall within the square with vertices at $13(1+i)$ and $16(1+i)$. Figure 7 displays within that square the null curves of the real and imaginary parts of the error function. Superimposed upon that display are red dots derived from (10.1), blue dots derived fom (10.2). For given values of $\{x,\nu\}$ the function $Z(x,\nu,b)$ inscribes b-parameterized straight lines on the complex plane; those lines are shown in their respective colors.

The abrupt onset of extreme rapid oscillation displayed in Figure 6 is seen thus to result when the increasing value of b has moved $Z(x,\nu,b)$ off the flat unit plane into the region of tormented soaring mountains.

At (9.0), p-values that fall outside the interval $[-2,+2\,]$ are explicitly excluded by the limits placed on the integral. The Gaussian simplicity of the integrand permitted analytic evaluation of the truncated integral, producing (9.1), where the factor $\mathcal{E}(x,b,\nu)$ has been seen—by the mechanism just described (Figure 7)—to impose (Figures 6) the condition $0 < b \leqslant 2$, giving finally

$$\mathbb{B}_{\text{band-limited}}(x,\nu,b) = \mathbb{B}(x,\nu,b) \quad : \quad 0 < b \leqslant b_{\max} = 2 \tag{11}$$

where $\mathbb{B}(x,\nu,b)$ can by manipulation of (8) be described

$$\begin{aligned}\mathbb{B}(x,\nu,b) = \frac{1}{(1+\xi^2)^{\frac{1}{4}}} &\exp\Big[\frac{b^2 x\xi}{2(1+\xi^2)}\Big]\\ &\times \exp\Big[i\,x\Big(1+\frac{\frac{1}{2}b^2}{(1+\xi^2)}\Big) - i\,\tfrac{1}{2}\text{arctan}\,\xi\Big] \quad : \quad \xi = x/2\nu\end{aligned}$$

Figures 8 display superimposed graphs of $\cos\big[x\left(1+\frac{1}{2}b^2\right)\big]$ and $\Re\big[\mathbb{B}(x,\nu,b)\big]$ for assorted values of $\{\nu,b\}$. The point is that, while the figures are qualitativly similar, only those with $b \leqslant 2$ (meaning asymptotic wavenumber $K \leqslant 3$) can be considered to comprise illustrations of superoscillation.

Remarks. The preceding discussion—particularly as it referred to properties of the error function with complex argument—relied entirely/critically upon what *Mathematica* has to say graphically about that subject. One would like to be in position to develop all that material analytically, but so intricately complex is the subject that that would appear to be a very heavy undertaking.

Berry has directed our attention to a tractable Gaussian *model*. One might expect qualitatibly similar results to follow from equations like (9.0) when one adopts different functions $k(p)$, different representations of the complex δ-function, places differently dictated limits on the integral. But one expects such adjustments to lead beyond the bounds of analytic tractability, to require profoundly novel modes of analysis.

Asymptotic approximation by the method of steepest descent. With the exact analytic result (9) in hand there would appear to be no need to appeal to approximation methods. But in the more general cases mentioned just above such methods may provide the only available avenue. It is, therefore, of interest to learn what they have to contribute in the present simple instance.

The integral (9.0) can be written

$$\mathbb{B}_{\text{band-limited}}(x,b,\nu) = \sqrt{\nu/\pi}\int_{-2}^{2} e^{f(p\,;\,x,\,b,\,\nu)}dp$$

with

$$f(p;x,b,\nu) = -\nu(p-ib)^2 + i\,x(1-\tfrac{1}{2}p^2) \quad : \quad \text{abbreviated } f(p) \tag{12}$$

where $f(p)$ is analytic on the complex p-plane. Writing $p+iq$ to denote points on that plane, we have

$$f(p+iq) = u(p,q) + iv(p,q)$$

with

$$\begin{aligned} u(p,q) &= pqx + b^2\nu - p^2\nu - 2bq\nu + q^2\nu \\ v(p,q) &= x - \tfrac{1}{2}p^2x + \tfrac{1}{2}q^2x + 2bp\nu - 2pq\nu \end{aligned}$$

which, as we verify, do satisfy the Cauchy-Riemann conditions:

$$u_p = v_q \quad \text{and} \quad u_q = -v_p$$

To locate the saddlepoints (of which for given $\{x, b, \nu\}$ there is in the present instance only one, denoted s) we solve $u_p(p,q) = u_q(p,q) = 0$ (or equivalently $v_p(p,q) = v_q(p,q) = 0$) for p and q and obtain

$$\begin{aligned} s \equiv p_s + iq_s &= \frac{2bx\nu}{x^2+4\nu^2} + i\frac{4b\nu^2}{x^2+4\nu^2} \\ &= \frac{b}{\xi - i} \quad \text{if one sets } \nu = x/2\xi \end{aligned} \tag{13}$$

The resulting equations

$$p_s(\xi) = \frac{b\,\xi}{1+\xi^2} \quad \text{and} \quad q_s(\xi) = \frac{b}{1+\xi^2} \tag{14.1}$$

trace a ξ-parameterized curve $\mathcal{C}$ on the complex p-plane. Eliminating ξ between those equations, we obtain an implicit construction of $\mathcal{C}$:

$$p_s^2 = q_s(b - q_s) \quad \text{equivalently} \quad p_s^2 + (q_s - \tfrac{1}{2}b)^2 = (\tfrac{1}{2}b)^2 \tag{14.2}$$

Equations (14) provide alternative descriptions of a semi-circle (Figure 9), with center at $i\frac{1}{2}b$ and radius $\frac{1}{2}b$.

Expanding $f(p)$ about a saddlepoint s, we have[5]

$$f(p) = f(s) + f'(s)(p-s) + \tfrac{1}{2}f''(s)(p-s)^2$$

with

$$\begin{aligned} f(s) &= \frac{x^2b^2\nu}{x^2+4\nu^2} + ix\Big(1 + \frac{2b^2\nu^2}{x^2+4\nu^2}\Big) && (15.11) \\ &= \frac{xb^2\xi}{2(1+\xi^2)} + ix\Big(1 + \frac{b^2}{2(1+\xi^2)}\Big) && (15.12) \\ f'(s) &= 0 && (15.2) \\ f''(s) &= -2\nu - ix \\ &= -2\nu(1+i\xi) && (15.3) \end{aligned}$$

[5] Because $f(p)$ is in the present instance quadratic the series promptly terminates.

where the manipulative details have been entrusted to *Mathematica*. Equation (15.20) asserts simply that s is—by construction—a stationary point of the function $f(p+iq) = u(p,q) + iv(p,q)$. From the analyticity of $f(p+iq)$ it follows that $u(p,q)$ and $v(p,q)$ are harmonic, therefore that s marks the location not of a local extremum but (by default) of a saddlepoint. The 2^{nd}-derivative properties of $f(p+iq)$ at s, to which (15.3) allude, are exposed most clearly by the Hessian

$$\mathbb{H}(p+iq) = \begin{pmatrix} u_{pp}(p,q) & u_{pq}(p,q) \\ u_{qp}(p,q) & u_{qq}(p,q) \end{pmatrix}$$

which in the present instance reads

$$\mathbb{H}(s) = \begin{pmatrix} -2\nu & x \\ x & 2\nu \end{pmatrix}$$

This is a traceless real symmetric matrix (from which follow a host of familiar properties), with equal-and-opposite real eigenvalues

$$\lambda(s) = \pm\sqrt{x^2 + 4\nu^2} = \pm 2\nu\sqrt{1+\xi^2} = \pm|f''(s)|$$

We now have

$$\begin{aligned}\sqrt{\nu/\pi}\int_{-2}^{2} e^{f(p)}dp &= \sqrt{\nu/\pi}\int_{-2}^{2} e^{f(s)+f'(s)(p-s)+\frac{1}{2}f''(s)(p-s)^2}dp \\ &\approx \sqrt{\nu/\pi}\oint_C e^{\frac{1}{2}f''(s)(p-s)^2}dp \cdot e^{f(s)}\end{aligned}$$

where the deformed contour C passes through the saddlepoint in the direction indicated by the eigenvector associated with the negative eigenvalue (so as to achieve "steepest descent" as one moves away from the saddlepoint). From

$$\sqrt{\nu/\pi}\int_C e^{-\frac{1}{2}\lambda(s)(p-s)^2}dp = \sqrt{\nu/\pi}\sqrt{\frac{2\pi}{\lambda(s)}} = \sqrt{\frac{2\nu}{2\nu\sqrt{1+\xi^2}}} = \frac{1}{\sqrt{|1+i\xi|}}$$

we obtain finally—in steepest descent approximation ($\xi\downarrow 0$)—

$$\begin{aligned}\mathbb{B}_{\text{band-limited}}(x,b,\nu) &\approx \frac{1}{\sqrt{|1+i\xi|}}\,e^{f(s)} \\ &= \frac{1}{\sqrt{|1+i\xi|}}\exp\Big[\frac{xb^2\xi}{2(1+\xi^2)}\Big]\exp\Big[ix\Big(1+\frac{b^2}{2(1+\xi^2)}\Big)\Big] \qquad (16)\end{aligned}$$

This asymptotically approximated $\int_{-2}^{+2}$ integral differs from the exact $\int_{-\infty}^{+\infty}$ integral (8) in only this respect:

$$\frac{1}{\sqrt{1+i\xi}} = \frac{1}{\sqrt{|1+i\xi|}}e^{-i\frac{1}{2}\arctan\xi} \quad \text{has been replaced by} \quad \frac{1}{\sqrt{|1+i\xi|}}$$

It differs from the exact $\int_{-2}^{+2}$ integral (9.1) in that respect and additionally in the absence of the $\mathcal{E}(x,b,\nu)$-factor that served to impose upon b the superoscillation restriction $1 < b < 2$. REMARK: The $\mathcal{E}(x,b,\nu)$-factor entered into (9.1) as the price paid for truncation $\int_{-\infty}^{+\infty} \longrightarrow \int_{-2}^{+2}$ of a Gaussian integral. The formal appeal to the method of steepest descent that led to (16) is susceptible to the criticism that it failed to take that truncation into account. Berry has addressed that problem,[1] and has—by an "elementary argument" that I cannot yet claim to understand—recovered (without appeal to subtle properties of erf(complex argument)) the condition $b \leqslant 2$.

From (7) and (13); *i.e.* from $k(p) = 1 - \frac{1}{2}p^2$ and $s = b/(\xi - i)$, we obtain

$$k(s) = \underbrace{\Big(1 + \tfrac{1}{2}b^2 \frac{1-\xi^2}{(1+\xi^2)^2}\Big)}_{\mathcal{K}(b,\xi)} - i\,b^2 \frac{\xi}{(1+\xi^2)^2}$$

where $\mathcal{K}(b,\xi)$ is what Berry calls the "local wavenumber." Clearly

$$\begin{aligned}
\mathcal{K}(b,0) &= 1 + \tfrac{1}{2}b^2, \text{ which} = 3 \text{ when } b = b_{\max} = 2\\
\mathcal{K}(b,1) &= 1\\
\mathcal{K}(b,\xi) &< 1 \quad : \quad \xi > 1\\
\lim_{\xi\to\infty} \mathcal{K}(b,\xi) &= 1
\end{aligned}$$

—all of which is illustrated in Figure 10 (in which b has been assigned its maximal value $b_{\max} = 2$). Only for $0 < \xi < 1$ (*i.e.*, for $\nu > \frac{1}{2}x$) does the local wavenumber $\mathcal{K}(b,\xi)$ assume a (red) superoscillatory value > 1. The condition $0 < \xi < 1$ produces the red saddlepoints in Figure 9.

Concluding remarks. Generally, one expects results obtained by asymptotic analysis (such, for example, as the $\mathbb{A}(x,p,n \to \infty) = e^{iax}$ of page 2) to be valid only asymptotically. The results obtained by application of the method of steepest descent to Berry's model of band-limited superoscillation are in this respect exceptional. The method does fail to capture the complexities latent in the factor $\mathcal{E}(x,\nu,b)$ that appears in equations (9.1) and (9.2), but the net effect of those complexities (Figures 1–5) is to impose upon b the restriction $b \leqslant 2$, and the method of steepest descent does capture *all* features of the exact theory that conform to that restriction. This it manages to do because the model is in all relevant respects so Gaussian—the feature to which it owes its exceptional tractability.

Many of the functions that occur in the preceding text differ superficially from those that occur in Berry's paper for notational reasons[6] and because I have sometimes subjected the arguments of functions to manipulations intended to clarify their essential structure.

Berry's Figures 3 illustrate two characteristic features of his model. They are reproduced here as Figures 11, which serve to demonstrate that—notational

[6] The analog of Aharonov's parameter a is by Berry denoted A and has here been denoted b; my $\xi = x/2\nu$ is by Berry written $\xi = x\delta^2$.

differences notwithstanding—he and I are saying the same thing. His $A = 2$ and $\delta = 0.2$ have become my $b = 2$ and $\nu = 1/2\delta^2 = 12.5$. Superimposed upon a graph of $\cos 3x$ in Figure 11a are graphs of $\mathbb{B}(x, \nu, b)$ and data points derived from the exact function $\mathbb{B}(x, \nu, b)\mathcal{E}(x, \nu, b)$, where the construction of those functions is borrowed from Figures 8 & 6; x ranges from 0 to 6, and it is evident (compare Figures 8) that superoscillation persists only to $x \approx 1$. In Figure 11aa the exact function is displayed as a dashed red curve. The superposition is precise because for $b \leqslant 2$ the $\mathcal{E}$-factor is unity. Figure 11b differs from Figure 11a only in that x ranges now from 40 to 46; the agreement is still precise (for the same reason as before) but the data points are now of order 10^{16}. Figure 11c displays the $\log_{10}$ of the absolute value of the real part of $\mathbb{B}(x, \nu, b)$. Beyond the small-x superoscillatory region the $\log_{10}$ grows linearly to large values, indicating exponential growth of $|\Re[\mathbb{B}(x, \nu, b)]|$. There are two spikes per period, and they become progressively more widely separated because

$$
\begin{aligned}
\text{local wavelength} = \frac{2\pi}{\text{local wavenumber}} &= 2\pi\Big[1 + \frac{b^2}{2(1 + x^2/4\nu^2)}\Big]^{-1} \\
&= \frac{2\pi(x^2 + 4\nu^2)}{x^2 + 2(2 + b^2)\nu^2}
\end{aligned}
$$

gives

$$
\begin{aligned}
\lim_{x \to 0} \text{local wavelength} &= \frac{4\pi}{2 + b^2} \\
&= 2\pi/3 \quad \text{at} \quad b = 2 \\
\lim_{x \to \infty} \text{local wavelength} &= 2\pi
\end{aligned}
$$

Gridlines at 0.55 and 2.70 bound a wavelength $\lambda = 2.15 = 1.032 \times 2\pi/3$, while gridlines at 21.25 and 27.35 bound a wavelength $\lambda = 6.10 = 0.971 \times 2\pi$; we expect local wavelength to expand by a factor of 3, and find that on the short compass of the figure it has expanded by a factor of 2.837.